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Fourier Approximation and Hausdorff Convergence[1]

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For a function f on $\mathbb{R}^d$ we consider its Fourier transform $\mathscr{F}f$ and the (integrable) Cesáro averages $\bar{\mathscr{F}}_M f$ of suitable truncations of $\mathscr{F}f$, described by the formula $(\bar{\mathscr{F}}_M f)(a) = M^{-d}(M-|a_1|)^+ \cdots (M-|a_d|)^+ (\mathscr{F}f)(a)$. We study the speed of the convergence $\mathscr{F}^{-1}\bar{\mathscr{F}}_M f \to f$, $(M \to \infty)$, under a metric that is somewhere between the L^1- and the L^∞-metrics. In this metric (which is appropriate to problems of pattern recognition), the distance between two functions is, more or less, the Hausdorff distance between their graphs. We describe a class of functions f for which the distance between $\mathscr{F}^{-1}\bar{\mathscr{F}}_M f$ and f is $\mathcal{O}(M^{-1/2})$, the fastest rate of converges one can have for discontinuous f.

1. INTRODUCTION

For an integrable function f on $\mathbb{R}^d$ we define its Fourier transform, $\mathscr{F}f$, and its inverse Fourier transform, $\mathscr{F}^{-1}f$, by:

$$(\mathscr{F}f)(a) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)\, e^{i\langle x, a\rangle}\, dx,$$

$$(a \in \mathbb{R}).$$

$$(\mathscr{F}^{-1}f)(a) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)\, e^{-i\langle x, a\rangle}\, dx$$

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If $\mathscr{F}f$ is integrable, f may be recuperated from $\mathscr{F}f$ by the formula $f = \mathscr{F}^{-1}\mathscr{F}f$. In general, $\mathscr{F}^{-1}\mathscr{F}f$ does not exist, but one may consider the truncations $\mathscr{F}_m f$ where $m = (m_1, ..., m_d) \in (0, \infty)^d$, defined by

$$(\mathscr{F}_m f)(a) := \begin{cases} (\mathscr{F}f)(a) & \text{if } \ |a_1| \leqslant m_1, ..., |a_d| \leqslant m_d, \\ 0 & \text{if not,} \end{cases}$$

and their Cesàro averages $\bar{\bar{\mathscr{F}}}_M f (M > 0)$:

$$(\bar{\bar{\mathscr{F}}}_M f)(a) = M^{-d} \int_0^M \cdots \int_0^M (\mathscr{F}_m f)(a)\, dm_1 \cdots dm_d.$$

One then has $\mathscr{F}^{-1}\bar{\bar{\mathscr{F}}}_M f \to f (M \to \infty)$ in the sense of $L^1(\mathbb{R}^d)$.

Our goal is to study the speed of convergence, where, however, we do not use the L^1-metric but a less conventional distance concept that somewhat resembles the Skorohod metric.

The idea behind this approach originates in problems of pattern recognition. There, typically, f is the indicator of a set X that one wants to reconstruct (approximately) from $\mathscr{F}f$ or its truncations, $\mathscr{F}_m f$. The L^1-approximation mentioned above is sometimes too rough, as $\mathscr{F}_m f$ tends to ignore small pieces of X. In such cases, one would prefer uniform approximation, but $\mathscr{F}^{-1}\mathscr{F}_m f$ and $\mathscr{F}^{-1}\bar{\bar{\mathscr{F}}}_M f$ are continuous and therefore cannot converge uniformly to f. What one can do is consider the graph of $\mathscr{F}^{-1}\bar{\bar{\mathscr{F}}}_M f$ (in $\mathbb{R}^d \times \mathbb{R}$) and observe that, for large M, it closely resembles the set $\bar{\Gamma}_f$ obtained from the graph of f by adding all points (x, t) where x lies in the boundary of the set X and t lies in the interval $[0, 1]$. If X is not too wild, for sufficiently large M every point of the graph of $\mathscr{F}^{-1}\bar{\bar{\mathscr{F}}}_M f$ is close to $\bar{\Gamma}_f$, and vice versa. (Owing to the Gibbs phenomenon, $\mathscr{F}^{-1}\mathscr{F}_m f$ instead of $\mathscr{F}^{-1}\bar{\bar{\mathscr{F}}}_M f$ will not do.) This idea leads to the introduction of a metric on a certain class of closed subsets of $\mathbb{R}^d \times \mathbb{R}$, analogous to the Hausdorff metric on the class of all compact subsets. From this metric we obtain a pseudometric $d_{\mathscr{H}}$ on the set of all bounded functions on $\mathbb{R}^d$. To some extent, this $d_{\mathscr{H}}$ behaves like the uniform metric (indeed, it is equivalent to the uniform metric on the set of all uniformly continuous bounded functions). On the other hand, in the sense of $d_{\mathscr{H}}$ the functions $\mathscr{F}^{-1}\bar{\bar{\mathscr{F}}}_M f$ tend to f if, say, f is the indicator of a bounded regularly closed set.

Our main purpose is to study speed of convergence. In Theorem 4.2 we describe a large class of functions f for which $d_{\mathscr{H}}(\mathscr{F}^{-1}\bar{\bar{\mathscr{F}}}_M f, f) = \mathcal{O}(M^{-1/2})$ $(M \to \infty)$. At the end of the paper we show that, for indicators of nonempty bounded sets, one cannot have faster convergence.

2. THE PSEUDOMETRIC

Let f be a bounded function on $\mathbb{R}^d$. We denote its graph by Γ_f and define

$$\begin{aligned} \Gamma_{f,l} &:= \{(x, t) \in \mathbb{R}^d \times \mathbb{R} : t \leqslant f(x)\}, \\ \Gamma_{f,u} &:= \{(x, t) \in \mathbb{R}^d \times \mathbb{R} : t \geqslant f(x)\}. \end{aligned} \tag{2.1}$$

(These sets are sometimes called the *hypograph* and the *epigraph* of f.) The *extended graph* of f,

$$\bar{\Gamma}_f, \tag{2.2}$$

is the intersection of the closures of $\Gamma_{f,l}$ and $\Gamma_{f,u}$. It is also the boundary of $\Gamma_{f,l}$ and of $\Gamma_{f,u}$. It is a closed subset of $\mathbb{R}^d$, containing the graph of f. If f happens to be continuous, $\bar{\Gamma}_f$ is the graph of f. If f is the indicator of a set $X \subset \mathbb{R}^d$, then

$$\bar{\Gamma}_f = \Gamma_f \cup (\partial X \times [0, 1]), \tag{2.3}$$

∂X being the boundary of X.

For two bounded functions, f and g, on $\mathbb{R}^d$ we define

$$d_{\mathscr{H}}(f, g) \tag{2.4}$$

to be the Hausdorff distance of $\bar{\Gamma}_f$ and $\bar{\Gamma}_g$, i.e., the infimum of all positive numbers r with the property

$$\begin{aligned} z \in \bar{\Gamma}_f &\Rightarrow \text{there is a } w \in \bar{\Gamma}_g \text{ with } \|z - w\| < r, \\ w \in \bar{\Gamma}_g &\Rightarrow \text{there is a } w \in \bar{\Gamma}_f \text{ with } \|z - w\| < r, \end{aligned} \tag{2.5}$$

obtaining a pseudometric $d_{\mathscr{H}}$ on the set of all bounded functions.

It is clear that uniform convergence implies $d_{\mathscr{H}}$-convergence. For uniformly continuous functions, the converse is also true. In fact, we have:

LEMMA 2.1. *Let $f_1, f_2, \ldots$ be bounded functions on $\mathbb{R}^d$, let $f: \mathbb{R}^d \to \mathbb{R}$ be bounded and uniformly continuous. Then*

$$d_{\mathscr{H}}(f_n, f) \to 0 \quad \Leftrightarrow \quad f_n \to f \text{ uniformly.}$$

Proof. One implication is obvious, as $d_{\mathscr{H}}(f_n, f) \leqslant \|f_n - f\|_\infty$ for all n. Now assume $d_{\mathscr{H}}(f_n, f) \to 0$. Let $\varepsilon > 0$. Choose $\delta \in (0, \varepsilon/2]$ such that $|f(x) - f(y)| \leqslant \varepsilon/2$ as soon as $x, y \in \mathbb{R}^d$, $\|x - y\| \leqslant \delta$. For sufficiently large n, we have $d_{\mathscr{H}}(f_n, f) < \delta$. For such n and for any $a \in \mathbb{R}^d$ there is a $b \in \mathbb{R}^d$ with $\|(a, f_n(a)) - (b, f(b))\| < \delta$; then certainly $\|a - b\| < \delta$ and $|f_n(a) - f(b)| < \delta$, whence $|f(a) - f(b)| < \varepsilon/2$ and $|f_n(a) - f(a)| < \delta + \varepsilon/2 \leqslant \varepsilon$. ∎

For practically obtaining estimates of $d_{\mathscr{H}}(f, g)$ the following observation is occasionally useful. Write

$$\bar{B}^{(d+1)} := \{(x, \lambda) \in \mathbb{R}^d \times \mathbb{R} : \|x\| \leqslant 1, |\lambda| \leqslant 1\}$$

and for $r > 0$ put $r\bar{B}^{(d+1)} := \{rz : z \in \bar{B}^{(d+1)}\}$. If f and g are bounded functions on $\mathbb{R}^d$ and $r > 0$, then

$$\begin{aligned} &d_{\mathscr{H}}(f, g) < r \quad \Rightarrow \quad \bar{\Gamma}_f \subset \bar{\Gamma}_g + r\bar{B}^{(d+1)} \text{ and } \bar{\Gamma}_g \subset \bar{\Gamma}_f + r\bar{B}^{(d+1)}, \\ &\bar{\Gamma}_f \subset \bar{\Gamma}_g + r\bar{B}^{(d+1)} \text{ and } \bar{\Gamma}_g \subset \bar{\Gamma}_f + r\bar{B}^{d+1} \quad \Rightarrow \quad d_{\mathscr{H}}(f, g) \leqslant 2r. \end{aligned} \tag{2.6}$$

For our results on $d_{\mathscr{H}}$-convergence of $\mathscr{F}^{-1}\bar{\mathscr{F}}_M f$ to f we need the following two facts.

LEMMA 2.2. *Let f be a bounded function on $\mathbb{R}^d$; let $a \in \mathbb{R}^d$, $r > 0$ and put*

$$\lambda := \inf_{\|x-a\| < r} f(x), \qquad \mu := \sup_{\|x-a\| < r} f(x).$$

Then for every $t \in (\lambda - r, \mu + r)$ we have

$$(a, t) \in \bar{\Gamma}_f + r\bar{B}^{(d+1)}.$$

Proof. Put $B_r(a) := \{x \in \mathbb{R}^d : \|x - a\| < r\}$. Let $t \in (\lambda - r, \mu + r)$. Choose a number s in the interval $[t - r, t + r] \cap (\lambda, \mu)$. Then the sets $X_- := \{x \in B_r(a) : f(x) \leqslant s\}$ and $X_+ := \{x \in B_r(a) : f(x) \geqslant s\}$ both are nonempty. There exists a point z of $B_r(a)$ that lies in the closure of both X_- and X_+. Choose a sequence $x_1, x_2, \ldots$ in X_-, converging to z and such that $s_- := \lim f(x_n)$ exists. Then $s_- \leqslant s$ and (z, s_-) lies in the closure of Γ_f, hence in $\bar{\Gamma}_f$. Similarly, there is an $s_+ \geqslant s$ with $(z, s_+) \in \bar{\Gamma}_f$. Thus, $(z, s) \in \bar{\Gamma}_f$ and

$$(a, t) \in (z, s) + r\bar{B}^{(d+1)} \subset \bar{\Gamma}_f + r\bar{B}^{(d+1)}. \quad \blacksquare$$

LEMMA 2.3. *Let f, g be bounded functions on $\mathbb{R}^d$. Let $r > 0$ and*

$$\Gamma_f \subset \bar{\Gamma}_g + r\bar{B}^{(d+1)}.$$

Then

$$\bar{\Gamma}_f \subset \bar{\Gamma}_g + r\bar{B}^{(d+1)}.$$

Proof. Let $(a, t) \in \bar{\Gamma}_f$. First, take any $\varepsilon > 0$. Put

$$\lambda := \inf\{g(z): z \in B_{r+2\varepsilon}(a)\}, \qquad \mu := \sup\{g(z): z \in B_{r+2\varepsilon}(a)\},$$

$B_{r+2\varepsilon}(a)$ being the open ball in $\mathbb{R}^d$ with center at a and radius $r + 2\varepsilon$.

(a, t) lies in the closure of $\Gamma_{f,l}$, so there is an $x \in \mathbb{R}^d$ with $\|x - a\| < \varepsilon$, $t < f(x) + \varepsilon$. There is a $(y, s) \in \bar{\Gamma}_g$ such that $\|x - y\| \leqslant r$ and $|f(x) - s| \leqslant r$. As (y, s) lies in the closure of $\Gamma_{g,l}$, there is a $z \in \mathbb{R}^d$ with $\|y - z\| < \varepsilon$, $s < g(z) + \varepsilon$. Then $\|z - a\| < r + 2\varepsilon$ and $t < g(z) + r + 2\varepsilon \leqslant \mu + r + 2\varepsilon$.

Similarly, $t > \lambda - r - 2\varepsilon$. Thus, by Lemma 2.2,

$$(a, t) \in \bar{\Gamma}_g + (r + 2\varepsilon)\, \bar{B}^{(d+1)}.$$

The above formula is valid for all $\varepsilon > 0$. Then, by compactness and because $\bar{\Gamma}_g$ is closed,

$$(a, t) \in \bar{\Gamma}_g + r\bar{B}^{(d+1)}. \quad \blacksquare$$

3. THE PROBLEM

Let f be an integrable function on $\mathbb{R}$ and let $\mathscr{F}$, $\mathscr{F}^{-1}$, $\mathscr{F}_m$, $\bar{\bar{\mathscr{F}}}_M$ be as in the Introduction. Then for all $M > 0$ and $a \in \mathbb{R}^d$ we obtain

$$(\mathscr{F}^{-1}\bar{\bar{\mathscr{F}}}_M f)(a) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)\, e^{i\langle x, t\rangle} \times \prod_{k=1}^{d} \left(1 - \frac{|t_k|}{M}\right)^+ e^{-i\langle a, t\rangle}\, dx\, dt. \tag{3.1}$$

The substitution $x = a + \frac{y}{M}$, $t = Ms$ yields

$$(\mathscr{F}^{-1}\bar{\bar{\mathscr{F}}}_M f)(a) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f\left(a + \frac{y}{M}\right) \times \prod_{k=1}^{d} (1 - |s_k|)^+\, e^{i\langle y, s\rangle}\, dy\, ds. \tag{3.2}$$

Thus,

$$(\mathscr{F}^{-1}\bar{\bar{\mathscr{F}}}_M f)(a) = \int_{\mathbb{R}^d} f\left(a + \frac{x}{M}\right) S(x)\, dx, \tag{3.3}$$

where, for all $x \in \mathbb{R}^d$,

$$S(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} \prod_{k=1}^{d} (1-|s_k|)^+ \, e^{i\langle x, s\rangle} \, ds$$
$$= (2\pi)^{-d} \prod_{k=1}^{d} \int_{\mathbb{R}} (1-|s_k|)^+ \, e^{ix_k s_k} \, ds_k$$
$$= \prod_{k=1}^{d} \sigma(x_k), \tag{3.4}$$

the function σ: $\mathbb{R} \to \mathbb{R}$ being defined by

$$\sigma(\xi) := \begin{cases} \dfrac{1}{\pi} \dfrac{1-\cos\xi}{\xi^2} & \text{if} \quad \xi \neq 0, \\[2ex] \dfrac{1}{2\pi} & \text{if} \quad \xi = 0. \end{cases} \tag{3.5}$$

Formula (3.4) determines a continuous nowhere negative function S on $\mathbb{R}^d$ whose integral is 1.

The subject of our investigation is convergence of $\mathscr{F}^{-1}\bar{\bar{\mathscr{F}}}_M f$ to f for bounded integrable functions f. A glance at (3.3), however, shows that it makes good sense to put the problem more generally. In fact, for any bounded measurable function f on $\mathbb{R}^d$, not necessarily integrable, we define functions f_M, $M > 0$, by

$$f_M(a) := \int_{\mathbb{R}^d} f\left(a + \frac{x}{M}\right) S(x) \, dx, \qquad a \in \mathbb{R}^d \tag{3.6}$$

and we will consider convergence of f_M to f as M tends to infinity.

We will frequently use the following estimates.

LEMMA 3.1.

$$0 \leqslant S(x) \leqslant (2\pi)^{-d}, \qquad x \in \mathbb{R}^d. \tag{3.7}$$

$$\int_{\|x\| \geqslant r} S(x) \, ds \leqslant \frac{2d^2}{r}, \qquad r > 0. \tag{3.8}$$

Proof of (3.8). If $x = (x_1, ..., x_d) \in \mathbb{R}^d$ and $|x_k| \leqslant r/d$ for $k = 1, ..., d$, then $\|x\| \leqslant r$. Hence, in the terminology of (3.4) and (3.5),

$$
\begin{aligned}
1-\int_{\|x\|\geqslant r} S(x)\,dx &= \int_{\|x\|\leqslant r}\left(\prod_{k=1}^{d}\sigma(x_k)\right)dx\\
&\geqslant \left(\int_{-r/d}^{r/d}\sigma(\xi)\,d\xi\right)^d\\
&= \left(1-2\int_{r/d}^{\infty}\sigma(\xi)\,d\xi\right)^d\\
&\geqslant 1-d\cdot 2\int_{r/d}^{\infty}\sigma(\xi)\,d\xi\\
&\geqslant 1-d\cdot 2\int_{r/d}^{\infty}\frac{2}{\pi\xi^2}\,d\xi\\
&= 1-\frac{4d^2}{\pi r}\geqslant 1-\frac{2d^2}{r}. \quad \blacksquare
\end{aligned}
\tag{3.9}
$$

For a given function f, in order to show that $d_{\mathscr{H}}(f_M, f)\to 0$ $(M\to\infty)$, one has to find functions α and β on $(0, \infty)$ such that $\alpha(M)\to 0$ and $\beta(M)\to 0$ $(M\to\infty)$, whereas, for all M,

$$\bar{\Gamma}_{f_M}\subset\bar{\Gamma}_f+\alpha(M)\,\bar{B}^{(d+1)},\qquad \bar{\Gamma}_f\subset\bar{\Gamma}_{f_M}+\beta(M)\,\bar{B}^{(d+1)}.$$

Curiously, for α we need no condition on f (except for boundedness and measurability). In fact, one may always take $\alpha(M):=d(\Delta f)^{1/2}M^{-1/2}$, where

$$\Delta f:=\sup_{x,\,y\in\mathbb{R}^d}|f(x)-f(y)|.$$

LEMMA 3.2. *Let f be a bounded measurable function on $\mathbb{R}^d$. Then*

$$\bar{\Gamma}_{f_M}\subset\bar{\Gamma}_f+d\sqrt{\Delta f}\cdot M^{-1/2}\bar{B}^{(d+1)}\qquad (M>0).$$

Proof. Put $K:=d\sqrt{\Delta f}$. Without restriction, assume that all values of f lie in the interval $[-\frac{1}{2}\Delta f, \frac{1}{2}\Delta f]$. Let $M>0$.

Take $a\in\mathbb{R}^d$. Putting

$$\mu:=\sup\{f(x)\colon \|x-a\|<KM^{-1/2}\}$$

we have $f(a+xM^{-1}) \leqslant \mu$ if $\|x\| < KM^{1/2}$ and $f(a+xM^{-1}) \leqslant \frac{1}{2}\Delta f$ for all x. Thus, with (3.8),

$$f_M(a) \leqslant \int_{\|x\| < KM^{1/2}} \mu S(x)\,dx + \int_{\|x\| \geqslant kM^{1/2}} \left(\frac{1}{2}\Delta f\right) S(x)\,dx$$
$$< \mu + \frac{1}{2}\Delta f \frac{2d^2}{KM^{1/2}} = \mu + KM^{-1/2}.$$

Similarly,

$$f_M(a) > \lambda - KM^{-1/2},$$

where

$$\lambda := \inf\{f(x) : \|x-a\| < KM^{-1/2}\}.$$

By applying Lemma 2.2 and observing that f_M is continuous, we find

$$\bar{\Gamma}_{f_M} = \Gamma_{f_M} \subset \bar{\Gamma}_f + KM^{-1/2}\bar{B}^{(d+1)}. \quad \blacksquare$$

We have $d_{\mathscr{H}}(f_M, f) \leqslant \|f_M - f\|_\infty \to 0$ $(M \to \infty)$ if f is (bounded and) uniformly continuous. Simple continuity is not sufficient, as is apparent from a rapidly oscillating function such as $x \mapsto \sin x^2 (x \in \mathbb{R})$.

The next theorem describes the functions f vanishing at infinity for which $d_{\mathscr{H}}(f_M, f) \to 0$ $(M \to \infty)$, but without an estimate for the speed of convergence. (An example is the indicator of a bounded regular open set.)

THEOREM 3.3. *Let $f: \mathbb{R}^d \to \mathbb{R}$ be bounded and measurable and such that $\lim_{\|x\| \to \infty} f(x) = 0$. Then the following conditions are equivalent.*

(i) *If $a \in \mathbb{R}^d$ and $\alpha, \beta \in \mathbb{R}$ are so that $\alpha \leqslant f \leqslant \beta$ a.e. on some neighborhood of a, then $\alpha \leqslant f(a) \leqslant \beta$.*

(ii) $\lim_{M \to \infty} d_{\mathscr{H}}(f_M, f) = 0$.

Proof. (i) ⇒ (ii). Let $\varepsilon > 0$. In view of Theorem 3.2, Formula (2.6) and Lemma 2.3 we only have to show that

$$\Gamma_f \subset \Gamma_{f_M} + \varepsilon \bar{B}^{(d+1)} \qquad \text{if } M \text{ is large enough.} \tag{$*$}$$

Choose $R > 0$ such that $\|x\| \leqslant R$ for every x with $|f(x)| \geqslant \frac{1}{4}\varepsilon$.

If $a \in \mathbb{R}^d$ and $\|a\| > R+1$, then $|f(a+y)| \leqslant \frac{1}{4}\varepsilon$ as soon as $\|y\| \leqslant 1$, so that for all M

$$|f_M(a)| \leqslant \frac{1}{4}\varepsilon + \int_{\|x\| \geqslant M} \left| f\left(a + \frac{x}{M}\right)\right| S(x)\,dx \leqslant \frac{1}{4}\varepsilon + \|f\|_\infty \frac{2d^2}{M},$$

whence

$$|f_M(a)-f(a)| \leqslant \frac{1}{4}\varepsilon + \|f\|_\infty \frac{2d^2}{M} + \frac{1}{4}\varepsilon.$$

It follows that for $M \geqslant 4\,\|f\|_\infty\, 2d^2\varepsilon^{-1}$:

$$\{(a, f(a)) : \|a\| > R+1\} \subset \Gamma_{f_M} + \varepsilon \bar{B}^{(d+1)}. \tag{3.10}$$

Cover $\{a \in \mathbb{R}^d : \|a\| \leqslant R+1\}$ by finitely many open sets $U_1, ..., U_N$ with diameters smaller than ε. For $n = 1, ..., N$ let

$$s_n := \inf\{s : f \leqslant s \text{ a.e. on } U_n\}.$$

Let X be the Lebesgue set of f. By a d-dimensional version of [1], 6.C, $f_M \to f$ pointwise on X. Furthermore, the complement of X is negligible. Thus, for each n we can choose an x_n in $U_n \cap X$ with $f(x_n) > s_n - \varepsilon/2$, and there is an M_1 such that $f_M(x_n) > s_n - \varepsilon$ for all $M \geqslant M_1$ and $n \in \{1, ..., N\}$. If $a \in U_n$ and $M \geqslant M_1$, then (using (i)):

$$f(a) \leqslant s_n < f_M(x_n) + \varepsilon.$$

Thus,

if $\|a\| \leqslant R+1$ and $M \geqslant M_1$, then $f(a) < \sup\{f_M(x) \colon \|x-a\| < \varepsilon\} + \varepsilon$.

Similarly, there is an M_2 for which

if $\|a\| \leqslant R+1$ and $M \geqslant M_2$, then $f(a) > \inf\{f_M(x) \colon \|x-a\| < \varepsilon\} - \varepsilon$.

If follows from Lemma 2.2 that for all sufficiently large M:

$$\{(a, f(a)) : \|a\| \leqslant R+1\} \subset \Gamma_{f_M} + \varepsilon \bar{B}^{(d+1)}. \tag{3.11}$$

Together with (3.10), this is $(*)$.

(ii) $\Rightarrow$ (i). Suppose $a \in \mathbb{R}^d$, $r > 0$, $\beta \in \mathbb{R}$, and $f \leqslant \beta$ a.e. on $B_{2r}(a)$. Let $\varepsilon > 0$; it suffices to prove $f(a) \leqslant \beta + 2\varepsilon$. Choose M so that $d_{\mathscr{H}}(f_M, f) < \min\{r, \varepsilon\}$ and $2d^2\,\|f\|_\infty < Mr\varepsilon$. Then there is a $b \in B_r(a)$ with $|f_M(b) - f(a)| < \varepsilon$. As $f \leqslant \beta$ a.e. on $B_r(b)$, we have

$$f_M(b) \leqslant \int_{\|x\| < Mr} \beta S(x)\,dx + \int_{\|x\| \geqslant Mr} \|f\|_\infty\, S(x)\,dx \leqslant \beta + \varepsilon,$$

whence $f(a) \leqslant \beta + 2\varepsilon$. ∎

4. THE CONDITION ON f

A function f on $\mathbb{R}^d$ is said to be *uniformly locally Lipschitz* if the following is true:

> There exist positive numbers h, α, C with the following property: for every $a \in \mathbb{R}^d$ there is a circular cone Σ_a with vertex a, height h and angle α, for which $|f(x)-f(y)| \leqslant C \|x-y\|$, $x, y \in \Sigma_a$. (4.1)

LEMMA 4.1. *Every uniformly locally Lipschitz function on $\mathbb{R}^d$ is Lebesgue measurable.*

Proof. Let $f: \mathbb{R}^d \to \mathbb{R}$ be uniformly locally Lipschitz. Let $\mathscr{A}$ be a maximal disjoint collection of subsets A of $\mathbb{R}^d$ that have the properties:

(1) A is measurable and has positive Lebesgue measure;

(2) the restriction of f to A is continuous.

Let X be the complement of the union of $\mathscr{A}$. It follows from (1) that $\mathscr{A}$ is countable, so X is measurable and (by (2)) we are done if it is negligible. Suppose it is not. Choose a density point a of X. There is a circular cone Σ, containing a, and such that the restriction of f to Σ is Lipschitz. Choose $c \in \mathbb{R}^d$ and $r > 0$ such that the ball $B_r(a+c)$ is contained in Σ. By the convexity of Σ, for every $\varepsilon \in (0, 1]$ we have $B_{\varepsilon r}(a+\varepsilon c) \subset \Sigma$.

As a is a density point of X, for some $\varepsilon \in (0, 1]$ we have (m being Lebesgue measure):

$$\begin{aligned} m(B_{\varepsilon r+\varepsilon\|c\|}(a) \cap X) &> \left(1-\left(\frac{r}{r+\|c\|}\right)^d\right) m(B_{\varepsilon r+\varepsilon\|c\|}(a)) \\ &= m(B_{\varepsilon r+\varepsilon\|c\|}(a)) - m(B_{\varepsilon r}(a+\varepsilon c)) \\ &= m(B_{\varepsilon r+\varepsilon\|c\|}(a) - B_{\varepsilon r}(a+\varepsilon c)). \end{aligned}$$

This is possible only if $B_{\varepsilon r}(a+\varepsilon c) \cap X$ has positive measure. But $B_{\varepsilon r}(a+\varepsilon c) \subset \Sigma$, so f is Lipschitz on $B_{\varepsilon r}(a+\varepsilon c) \cap X$. This contradicts the maximality of $\mathscr{A}$. ▮

THEOREM 4.2. *Let $f: \mathbb{R}^d \to \mathbb{R}$ be bounded and uniformly locally Lipschitz. Then*

$$d_{\mathscr{H}}(f, f_M) = \mathcal{O}(M^{-1/2}), \qquad M \to \infty.$$

Proof. Take C, $\Sigma_a (a \in \mathbb{R}^p)$ as in (4.1). Write $\Sigma := \Sigma_0$; then every Σ_a is isometric to Σ. Let $r > 0$ be such that Σ contains a ball with radius r.

Because of Theorem 3.2 and Lemma 2.3, it is enough to find a number K with

$$\Gamma_f \subset \Gamma_{f_M} + KM^{-1/2}\bar{B}^{(d+1)} \qquad \text{if} \quad M > r^{-2}. \tag{$*$}$$

We show that this is satisfied by:

$$K := \max\{K_1, K_2\}, \qquad K_1 := C\left(\frac{\operatorname{diam}\Sigma}{r} + 1\right) + 2d^2\Delta f, \qquad K_2 := \frac{\operatorname{diam}\Sigma}{r}.$$

Take $a \in \mathbb{R}^d$ and $M > r^{-2}$. Choose $c \in \mathbb{R}^d$ so that $B_r(a+c) \subset \Sigma_a$. Setting $\delta := r^{-1}M^{-1/2}$ we have $0 < \delta < 1$, whence

$$B_{M^{-1/2}}(a+\delta c) = B_{\delta r}(a+\delta c) \subset \Sigma_a$$

and

$$\begin{aligned}
&|f(a) - f_M(a+\delta c)| \\
&\quad \leqslant \int_{x\in\mathbb{R}^d} \left| f(a) - f\left(a + \delta c + \frac{x}{M}\right)\right| S(x)\,dx \\
&\quad \leqslant \int_{\|x\| < M^{1/2}} C \left\| \delta c + \frac{x}{M} \right\| S(x)\,dx + \int_{\|x\| \geqslant M^{1/2}} (\Delta f)\, S(x)\,dx \\
&\quad \leqslant C\left(\delta\cdot\operatorname{diam}\Sigma + \frac{1}{\sqrt{M}}\right) + (\Delta f)\cdot\frac{2d^2}{\sqrt{M}} \\
&\quad = K_1 M^{-1/2}.
\end{aligned}$$

As $\|a - (a+\delta c)\| \leqslant \delta\cdot\operatorname{diam}\Sigma = K_2 M^{-1/2}$, we have ($*$). ∎

THEOREM 4.3. *Suppose $X_1, ..., X_N$ are subsets of $\mathbb{R}^d$ whose union is a ball B and assume that each X_n either is convex with nonempty interior or has a smooth boundary. Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be Lipschitz on each X_n and vanish identically off B. Then*

$$d_{\mathscr{H}}(f, f_M) = \mathcal{O}(M^{-1/2}), \qquad M \to \infty.$$

Proof. Let X_0 be the complement of B; then X_0 has a smooth boundary. We are done if for every $n \in \{0, 1, ..., N\}$ there exist h_n, $\alpha_n > 0$ with the property that every point of X_n is the vertex of a circular cone that has height h_n and angle α_n, and is contained in X_n. It follows readily from the proof of Theorem 5.3 in [4] that such h_n and α_n exist in case X_n has a smooth boundary. If X_n is convex and has nonempty interior, choose a closed ball $\bar{B}_r(c)$ in X_n and let R be the diameter of X_n; for every a in X_n

we see that X_n contains the convex hull of $\{a\} \cup \bar{B}(c)$, and thereby a cone with vertex a, height r and angle arc sin $R^{-1}r$. ∎

For indicator functions of bounded sets, Theorem 4.2 is optimal in the following sense. *Let X be a measurable subset of $\mathbb{R}^d$ that is bounded, nonempty, and let f be its indicator. Then*

$$d_{\mathscr{H}}(f, f_M) \geqslant \tfrac{1}{10} M^{-1/2}, \qquad M \in \mathbb{N}.$$

Proof. By translation invariance we may assume

$$X \subset [0, \infty) \times \mathbb{R}^{d-1}, \qquad \text{the closure of } X \text{ contains } 0.$$

Take $M \in \mathbb{N}$ and $\delta > d_{\mathscr{H}}(f, f_M)$. As $(0, 1)$ lies in the closure of Γ_f, there must exist an $a \in \mathbb{R}^d$ with $\|(0, 1) - (a, f_M(a))\| < \delta$; then certainly $\|a\| < \delta$ and $1 - f_M(a) < \delta$. If g is the indicator of $(0, \infty) \times \mathbb{R}^{d-1}$, then $g \leqslant 1 - f$, whence $g_M \leqslant 1 - f_M$. By (3.3), applied to g, one sees that

$$g_M(a) = \int_{-M\alpha}^{\infty} \sigma(\xi)\, d\xi,$$

where α is the first coordinate of a. As $-\alpha \leqslant \delta$, one has

$$\delta > 1 - f_M(a) \geqslant g_M(a) = \int_{-M\alpha}^{\infty} \sigma(\xi)\, d\xi \geqslant \int_{M\delta}^{\infty} \sigma(\xi)\, d\xi.$$

From this, it is not hard to obtain $M\delta^2 \geqslant 10^{-2}$, i.e., $\delta \geqslant 10^{-1} M^{-1/2}$.

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